DOCUMENT RESUME

ED 299 163 SE 049 718

AUTHOR Goodman-Petrushka, Sharon
TITLE Integration Review Workbook. Exercises with Hints and Completed Solutions on Indefinite Integration, as Taught in Calculus.
PUB DATE 88
NOTE 36p.
PUB TYPE Guides - Classroom Use - Materials (For Learner) (051)

EDRS PRICE MF01/PC02 Plus Postage.
DESCRIPTORS *Calculus; College Mathematics; *Drills (Practice); *Functions (Mathematics); Higher Education; Mathematics Curriculum; Mathematics Education; *Mathematics Instruction; Problem Solving; *Study; Trigonometry

ABSTRACT

This workbook was designed to be used as a study aid in any course covering the various techniques of indefinite integration. Many students are able to master each individual technique, but upon encountering an integral on an exam, they often have difficulties in determining which technique to use. By working through all of the exercices in this workbook, students will begin to overcome this problem. Each of the seven sections of this workbook contains a set of functions to integrate. Students should first attempt to work through each problem on their own. If uncertain how to begin the problem, they should refer to the hint and try again. Then they may look at the detailed solution to check their answers. (Author/PK)

****
* Reproductions supplied by EDRS are the best that can be made *
* from the original document. *
****

ED299163

INTEGRATION REVIEW WORKBOOK

Exercises with hints and completed solutions on indefinite integration, as taught in Calculus.

Dr. Sharon Goodman-Petrushka
Department of Mathematics
New York Institute of Technology
Old Westbury, New York

U.S. DEPARTMENT OF EDUCATION
Office of Educational Research and Improvement
EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC)

X This document has been reproduced as received from the person or organization originating it.

☐ Minor changes have been made to improve reproduction quality.

• Points of view or opinions stated in this document do not necessarily represent official OERI position or policy.

"PERMISSION TO REPRODUCE THIS MATERIAL HAS BEEN GRANTED BY

Sharon Goodman-Petrushka

TO THE EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC)."

SE 049 718

BEST COPY AVAILABLE

This workbook was designed to be used as a study aid in any course covering the various techniques of indefinite integration. Many students are able to master each individual technique, but upon encountering an integral on an exam, they often have difficulties in determining which technique to use. By working through all of the exercises in this workbook, the student will begin to overcome this problem.

Each of the seven sections of this workbook contains a set of functions to integrate which appear similar to each other, but which require different techniques of integration. The student should first attempt to work through each problem on his/her own. If uncertain how to begin the problem, he should refer to the hint and try again. Then he may look at the detailed solution to check his answer.

REVIEW SHEET # 1

1. $\int (4 - x^2)^3 \, dx$

2. $\int x \, (4 - x^2)^8 \, dx$

3. $\int \frac{dx}{\sqrt{4 - x^2}}$

4. $\int \frac{x \, dx}{\sqrt{4 - x^2}}$

5. $\int \frac{x \, dx}{\sqrt{4 - x^4}}$

6. $\int \frac{x^3 \, dx}{\sqrt{4 - x^2}}$

7. $\int \frac{dx}{x\sqrt{4 - x^2}}$

8. $\int \frac{dx}{\sqrt{4 - x^2 - 2x}}$

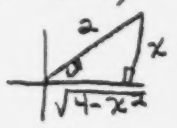
9. $\int \frac{(x + 1) \, dx}{\sqrt{4 - x^2 - 2x}}$

10. $\int \frac{(3x + 5) \, dx}{\sqrt{4 - x^2 - 2x}}$

REVIEW SHEET #1 - SOLUTIONS

Problem	Hint	Solution
1. $\int (4-x^2)^3\,dx$	Do we have the derivative of the parenthesis sitting next to it? Can we use the Power rule: $\int [f(x)]^n f'(x)\,dx = \frac{1}{n+1}[f(x)]^{n+1} + C$ $n \neq -1$?	We can't use the power rule. We must multiply out and integrate term by term: $\int (64 - 48x^2 + 12x^4 - x^6)\,dx$ $= 64x - 16x^3 + \frac{12}{5}x^5 - \frac{1}{7}x^7 + C$
2. $\int x(4-x^2)^8\,dx$	see above	This time we can use the power rule because the derivative of the parenthesis is sitting right there, except for the constant which we insert and correct with the boxed numbers: $\boxed{\frac{-1}{2}} \int (4-x^2)^8 \boxed{-2}\,x\,dx =$ $\frac{-1}{2}\cdot\frac{1}{9}(4-x^2)^9 + C =$ $\frac{-1}{18}(4-x^2)^9 + C$
3. $\int \frac{dx}{\sqrt{4-x^2}}$	see above	We don't have the derivative of $4-x^2$ in the numerator, so we can't use the power rule. We must recognize this integral as: $\sin^{-1}\frac{x}{2} + C$

Problem	Hint	Solution
4. $\int \frac{x}{\sqrt{4-x^2}}\,dx$	see above	As in problem #2, we can use the power rule: $\boxed{\frac{-1}{2}} \int (4-x^2)^{-1/2} \boxed{-2}\, x\,dx =$ $\frac{-1}{2} \cdot 2\,(4-x^2)^{1/2} + C =$ $-\sqrt{4-x^2} + C$
5. $\int \frac{x}{\sqrt{4-x^4}}\,dx$	Requires a trick: $x^4 = (x^2)^2$	We use the u-substitution: $u = x^2$ $du = 2x\,dx$; $\int \frac{x\,dx}{\sqrt{4-(x^2)^2}} = \boxed{\frac{1}{2}} \int \frac{\boxed{2}\,x\,dx}{\sqrt{4-(x^2)^2}} =$ $\frac{1}{2}\int \frac{du}{\sqrt{4-u^2}} = \frac{1}{2} \sin^{-1}\frac{u}{2} + C =$ $\frac{1}{2}\sin^{-1}\left(\frac{x^2}{2}\right) + C$
6. $\int \frac{x^3}{\sqrt{4-x^2}}\,dx$	$x^3 = x^2 \cdot x$	We use the u-substitution: $u = 4 - x^2$ $du = -2x\,dx$; $\int \frac{x^2\, x\,dx}{\sqrt{4-x^2}} =$ $\boxed{\frac{-1}{2}} \int x^2\,(4-x^2)^{-\frac{1}{2}} \boxed{-2}\, x\,dx =$ $\frac{-1}{2}\int (4-u)\,u^{-\frac{1}{2}}\,du =$ $\frac{-1}{2}\int (4u^{-\frac{1}{2}} - u^{\frac{1}{2}})\,du =$ (continued)

Problem	Hint	Solution
6. (continued)		$\frac{-1}{2}\left[4\,(2)\,u^{\frac{1}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right] + C =$ $-4\sqrt{4-x^2} + \frac{1}{3}\sqrt{(4-x^2)^3} + C$
7. $\int \frac{dx}{x\sqrt{4-x^2}}$	Is it a power rule? an inverse trig.,? It's similar to an inverse trig. formula, but not exactly !	We use trig. substitution: $x = 2\sin\theta\ ,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ $dx = 2\cos\theta\, d\theta$ $\sqrt{4-x^2} = \sqrt{4-4\sin^2\theta} =$ $\sqrt{4(1-\sin^2\theta)} = \sqrt{4\cos^2\theta} =$ $\lvert 2\cos\theta\rvert = 2\cos\theta$ and $\int \frac{dx}{x\sqrt{4-x^2}} = \int \frac{2\cos\theta\, d\theta}{(2\sin\theta)(2\cos\theta)}$ $= \frac{1}{2}\int \csc\theta\, d\theta$  $= \frac{1}{2}\ln\left\lvert \csc\theta - \cot\theta \right\rvert + C$ $= \frac{1}{2}\ln\left\lvert \frac{2}{x} - \frac{\sqrt{4-x^2}}{x} \right\rvert + C$
8. $\int \frac{dx}{\sqrt{4-x^2-2x}}$	Complete the square	$4-x^2-2x = 4+1-(x^2+2x+1) =$ $5-(x+1)^2$, so $\int \frac{dx}{\sqrt{5-(x+1)^2}} = \int \frac{du}{\sqrt{5-u^2}} =$ $\sin^{-1}\left(\frac{u}{\sqrt{5}}\right) + C = \sin^{-1}\left(\frac{x+1}{\sqrt{5}}\right) + C$ (where $u = x+1$ $du = dx$)

Problem	Hint	Solution
9. $\int \frac{(x+1)\,dx}{\sqrt{4-x^2-2x}}$	$\frac{d}{dx}\,(4-x^2-2x) = ?$	The Power rule works! $\boxed{\frac{-1}{2}} \int (4-x^2-2x)^{-\frac{1}{2}} \boxed{-2}\,(x+1)\,dx$ $= \frac{-1}{2}(2)(4-x^2-2x)^{\frac{1}{2}} + C$ $= -\sqrt{4-x^2-2x} + C$
10. $\int \frac{(3x+5)dx}{\sqrt{4-x^2-2x}}$	Complete the square	$\int \frac{(3x+5)dx}{\sqrt{5-(x+1)^2}} = \int \frac{(3u+2)\,du}{\sqrt{5-u^2}}$ [where $u = x+1$, $du = dx$, $x = u-1$, $3x+5 = 3u+2$] $= 3\int \frac{u\,du}{\sqrt{5-u^2}} + 2\int \frac{du}{\sqrt{5-u^2}}$ power rule $= \frac{3}{\boxed{-2}} \int \frac{\boxed{-2}\,u\,du}{\sqrt{5-u^2}} + 2\sin^{-1}\frac{u}{\sqrt{5}} + C$ $= -3\sqrt{5-u^2} + 2\sin^{-1}\frac{u}{\sqrt{5}} + C$ $= -3\sqrt{5-(x+1)^2} + 2\sin^{-1}\left(\frac{x+1}{\sqrt{5}}\right) + C$

REVIEW SHEET # 2

1. $\int \frac{dx}{5+x^2}$

2. $\int \frac{x\,dx}{5+x^2}$

3. $\int \frac{x^2\,dx}{5+x^2}$

4. $\int \frac{x^3\,dx}{5+x^2}$

5. $\int \frac{x\,dx}{(5+x^2)^3}$

6. $\int \sqrt{5+x^2}\,dx$

7. $\int x\sqrt{5+x^2}\,dx$

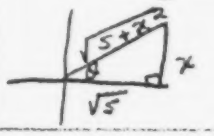
8. $\int x^3\sqrt{5+x^2}\,dx$

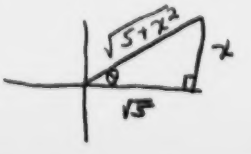
9. $\int \frac{dx}{(5+x^2)^2}$

REVIEW SHEET # 2 - SOLUTIONS

Problem	Hint	Solution
1. $\int \frac{dx}{5+x^2}$	Is it a power rule? Why ? This should be familiar.	$\frac{1}{\sqrt{5}} \tan^{-1} \frac{x}{\sqrt{5}} + C$
2. $\int \frac{x\,dx}{5+x^2}$	Is it a power rule? What is the power? $\int \frac{f'(x)\,dx}{f(x)} = \ln \lvert f(x) \rvert + C$	This one is a power rule: $\left[\frac{1}{2}\right] \int \frac{[2]\,x\,dx}{5+x^2}$ $= \frac{1}{2} \ln \lvert 5+x^2 \rvert + C$
3. $\int \frac{x^2\,dx}{5+x^2}$	For an improper rational function, what can we do?	We use long division: $\frac{x^2}{5+x^2} = 1 - \frac{5}{5+x^2}$ integrating this: $\int 1\,dx - \int \frac{5}{5+x^2}\,dx =$ $x - \frac{5}{\sqrt{5}} \tan^{-1} \frac{x}{\sqrt{5}} + C$
4. $\int \frac{x^3\,dx}{5+x^2}$	Use method of # 3 or $x^3 = x^2\,x$	Using long division: $\frac{x^3}{5+x^2} = x - \frac{5x}{5+x^2}$ Integrating this: $\int x\,dx - 5\int \frac{x\,dx}{5+x^2} =$ $\frac{x^2}{2} - 5\left[\frac{1}{2} \ln \lvert 5+x^2 \rvert\right] + C$ (see #2 above)

Problem	Hint	Solution
4. (continued)		Alternative solution: $\int \frac{x^2\ x\ dx}{5+x^2} = \boxed{\frac{1}{2}}\int \frac{x^2\ \boxed{2}\ x\ dx}{5+x^2}$ Using the u-substitution: $u = 5+x^2$, $du = 2x\ dx$, $x^2 = u-5$ we have $\frac{1}{2}\int \frac{u-5}{u}\ du = \frac{1}{2}\int (1 - \frac{5}{u})\ du$ $= \frac{1}{2}(u - 5\ln\lvert u\rvert) + C =$ $\frac{1}{2}(5+x^2) - \frac{5}{2}\ln\lvert 5+x^2\rvert + C$
5. $\int \frac{x\ dx}{(5+x^2)^3}$	Is it a power rule? Does your answer have an "ln"?	$\boxed{\frac{1}{2}}\int (5+x^2)^{-3}\boxed{2}\,x\ dx =$ $\frac{1}{2}\cdot\frac{1}{-2}(5+x^2)^{-2} + C =$ $-\frac{1}{4}\frac{1}{(5+x^2)^2} + C$
6. $\int\sqrt{5+x^2}\ dx$	Not exactly an inverse trig formula	We use trig. substitution: $x = \sqrt{5}\tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $dx = \sqrt{5}\sec^2\theta\ d\theta$ $\sqrt{5+x^2} = \sqrt{5+5\tan^2\theta} = \sqrt{5(1+\tan^2\theta)}$ $= \sqrt{5\sec^2\theta} = \sqrt{5}\sec\theta$ Thus we must integrate $\int\sqrt{5}\sec\theta\sqrt{5}\sec^2\theta\ d\theta =$ $5\int\sec^3\theta\ d\theta$ (continued)

Problem	Hint	Solution
6. (continued)		Using integration by parts: $u = \sec\theta$ $\quad dv = \sec^2\theta\, d\theta$ $du = \sec\theta\tan\theta\, d\theta$ $\quad v = \tan\theta$ We have: $5[\sec\theta\tan\theta - \int\sec\theta\tan^2\theta\, d\theta] =$ $5[\sec\theta\tan\theta - \int\sec\theta(\sec^2\theta - 1)\, d\theta] =$ $5\sec\theta\tan\theta - 5\int\sec^3\theta\, d\theta + 5\int\sec\theta\, d\theta$ so that $10\int\sec^3\theta\, d\theta = 5\sec\theta\tan\theta + 5\ln\lvert\sec\theta + \tan\theta\rvert$ and $\int\sqrt{5+x^2}\, dx = 5\int\sec^3\theta\, d\theta =$ $\frac{5}{2}\sec\theta\tan\theta + \frac{5}{2}\ln\lvert\sec\theta+\tan\theta\rvert + C$ $= \frac{5}{2}\frac{\sqrt{x^2+5}}{\sqrt{5}}\cdot\frac{x}{\sqrt{5}} + \frac{5}{2}\ln\left\lvert\frac{\sqrt{x^2+5}}{\sqrt{5}} + \frac{x}{\sqrt{5}}\right\rvert + C$ $= \frac{x\sqrt{x^2+5}}{2} + \frac{5}{2}\ln\left\lvert\frac{\sqrt{x^2+5}+x}{\sqrt{5}}\right\rvert + C$ 
7. $\int x\sqrt{5+x^2}\, dx$	$\frac{d}{dx}(5+x^2) = ?$	Power rule: $\frac{1}{2}\int(5+x^2)^{\frac{1}{2}}\,2x\, dx =$ $\frac{1}{2}\cdot\frac{2}{3}(5+x^2)^{\frac{3}{2}} + C = \frac{1}{3}(5+x^2)^{\frac{3}{2}} + C$

Problem	Hint	Solution
8. $\int x^3 \sqrt{5+x^2}\, dx$	$x^3 = x^2 \cdot x$	Using the u-substitution: $u = 5+x^2$ $du = 2x\, dx$ $x^2 = u-5$, we have $\boxed{\frac{1}{2}} \int x^2 \sqrt{5+x^2}\, \boxed{2}\, x\, dx =$ $\frac{1}{2}\int (u-5)\, u^{\frac{1}{2}}\, du =$ $\frac{1}{2}\left(u^{\frac{3}{2}} - 5u^{\frac{1}{2}}\right) du =$ $\frac{1}{2}\left[\frac{2}{5} u^{\frac{5}{2}} - 5\left(\frac{2}{3}\right) u^{\frac{3}{2}}\right] + C =$ $\frac{1}{5}(5+x^2)^{\frac{5}{2}} - \frac{5}{3}(5+x^2)^{\frac{3}{2}} + C$
9. $\int \frac{dx}{(5+x^2)^2}$	Similar to an inverse trig formula, but not exactly right !	We use the trig. substitution: $x = \sqrt{5}\tan\theta \quad , -\frac{\pi}{2} < \theta < \frac{\pi}{2}$ $dx = \sqrt{5}\sec^2\theta\, d\theta$ $(5+x^2)^2 = (5+5\tan^2\theta)^2 =$ $5(1+\tan^2\theta)^2 = (5\sec^2\theta)^2$ so we must integrate $\int \frac{\sqrt{5}\sec^2\theta\, d\theta}{25\sec^4\theta} = \frac{\sqrt{5}}{25}\int \cos^2\theta\, d\theta =$ $\frac{\sqrt{5}}{25}\int \frac{1}{2}(1+\cos 2\theta)\, d\theta =$ $\frac{\sqrt{5}}{50}(\theta + \frac{1}{2}\sin 2\theta) + C =$ $\frac{\sqrt{5}}{50}\theta + \frac{\sqrt{5}}{100} \cdot 2\sin\theta\cos\theta + C =$ $\frac{\sqrt{5}}{50}\tan^{-1}\frac{x}{\sqrt{5}} + \frac{\sqrt{5}}{50} \cdot \frac{x}{\sqrt{5+x^2}} \cdot \frac{\sqrt{5}}{\sqrt{5+x^2}} + C$ $= \frac{\sqrt{5}}{50}\tan^{-1}\frac{x}{\sqrt{5}} + \frac{1}{10}\frac{x}{5+x^2} + C$ 

REVIEW SHEET # 3

1. $\int \frac{e^{5x}}{9 + e^{5x}} \, dx$

2. $\int \frac{e^{5x}}{(9 + e^{5x})^7} \, dx$

3. $\int \frac{e^{5x}}{9 + e^{10x}} \, dx$

4. $\int \frac{e^{10x}}{9 + e^{5x}} \, dx$

5. $\int \frac{e^{5x}}{\sqrt{9 + e^{5x}}} \, dx$

6. $\int \frac{e^{5x}}{\sqrt{9 - e^{10x}}} \, dx$

7. $\int x^2 e^{5x} \, dx$

8. $\int e^{\frac{5}{x}} \cdot \frac{1}{x^2} \, dx$

REVIEW SHEET # 3 - SOLUTIONS

Problem	Hint	Solution
1. $\int \frac{e^{5x}}{9 + e^{5x}}\, dx$	What is the derivative of the denominator?	We use the power rule with $n = -1$: $\left[\frac{1}{5}\right]\int \frac{[5]\, e^{5x}}{9 + e^{5x}}\, dx =$ $\frac{1}{5} \ln \left\lvert 9 + e^{5x} \right\rvert + C$
2. $\int \frac{e^{5x}\, dx}{(9 + e^{5x})^7}$	Is this a power rule? Is it an "ln"?	We use the power rule with $n = -7$: $\left[\frac{1}{5}\right]\int (9 + e^{5x})^{-7} [5] e^{5x}\, dx =$ $\frac{1}{5} \cdot \frac{1}{-6} (9 + e^{5x})^{-6} + C =$ $-\frac{1}{30} (9 + e^{5x})^{-6} + C$
3. $\int \frac{e^{5x}\, dx}{9 + e^{10x}}$	$e^{10x} = (e^{5x})^2$	$\left[\frac{1}{5}\right]\int \frac{[5]\, e^{5x}\, dx}{9 + (e^{5x})^2} =$ $\begin{vmatrix} u = e^{5x} \\ du = 5\, e^{5x} dx \end{vmatrix}$ $\frac{1}{5}\int \frac{du}{9 + u^2} = \frac{1}{5} \cdot \frac{1}{3} \tan^{-1} \frac{u}{3} + C$ $= \frac{1}{15} \tan^{-1}\left[\frac{e^{5x}}{3}\right] + C$
4. $\int \frac{e^{10x}\, dx}{9 + e^{5x}}$	$e^{10x} = e^{5x} \cdot e^{5x}$	Using the u-substitution: $u = 9 + e^{5x}$ $du = 5\, e^{5x} dx$ $e^{5x} = u - 9$, we have (continued)

Problem	Hint	Solution
4. (continued)		$\left[\frac{1}{5}\right]\int \frac{e^{5x}\,\left[5\right]\,e^{5x}\,dx}{9+e^{5x}} =$ $\frac{1}{5}\int \frac{u-9}{u}\,du =$ $\frac{1}{5}\int\left[1-\frac{9}{u}\right]du = \frac{1}{5}\left[u-9\ln|u|\right]+C$ $= \frac{1}{5}(9+e^{5x}) - \frac{9}{5}\ln\left|9+e^{5x}\right| + C$
5. $\int \frac{e^{5x}\,dx}{\sqrt{9+e^{5x}}}$	See problem #2	Power rule with $n = \frac{1}{2}$: $\left[\frac{1}{5}\right]\int(9+e^{5x})^{-\frac{1}{2}}\left[5\right]e^{5x}\,dx =$ $\frac{1}{5}(2)(9+e^{5x})^{\frac{1}{2}} + C =$ $\frac{2}{5}\sqrt{9+e^{5x}} + C$
6. $\int \frac{e^{5x}\,dx}{\sqrt{9-e^{10x}}}$	$e^{10x} = (e^{5x})^2$.	Using the u-substitution: $u = e^{5x}$ $du = 5\,e^{5x}\,dx$, we have $\left[\frac{1}{5}\right]\int \frac{\left[5\right]\,e^{5x}\,dx}{\sqrt{9-(e^{5x})^2}} =$ $\frac{1}{5}\int \frac{du}{\sqrt{9-u^2}} = \frac{1}{5}\sin^{-1}\frac{u}{3} + C =$ $\frac{1}{5}\sin^{-1}\left(\frac{e^{5x}}{3}\right) + C$

Problem	Hint	Solution
7. $\int x^2 \, e^{5x} \, dx$	How do you often integrate a product?	We will have to use integration by parts twice: $u = x^2 \qquad dv = e^{5x} \, dx$ $du = 2x \, dx \qquad v = \frac{1}{5} e^{5x}$ $\int x^2 \, e^{5x} \, dx = \frac{1}{5} x^2 \, e^{5x} - \frac{2}{5} \int x \, e^{5x} dx$ For the last integral: $u = x \qquad dv = e^{5x} \, dx$ $du = dx \qquad v = \frac{1}{5} e^{5x}$ yielding $\frac{1}{5} x^2 \, e^{5x} - \frac{2}{5} \left[\frac{1}{5} x \, e^{5x} - \frac{1}{5} \int e^{5x} \, dx \right]$ $= \frac{1}{5} x^2 \, e^{5x} - \frac{2}{25} x \, e^{5x} + \frac{2}{25} \cdot \frac{1}{5} e^{5x} + C$ $= \frac{1}{5} x^2 \, e^{5x} - \frac{2}{25} x \, e^{5x} + \frac{2}{125} e^{5x} + C$
8. $\int e^{\frac{5}{x}} \cdot \frac{1}{x^2} \, dx$	Does this product require integration by parts?	A simple u-substitution will do: $u = \frac{5}{x} = 5 x^{-1}$ $du = -5 x^{-2} \, dx = \frac{-5}{x^2} \, dx$ Our integral then becomes $\left[-\frac{1}{5} \right] \int \frac{e^{\frac{5}{x}} \left[-5 \right] dx}{x^2} =$ $-\frac{1}{5} \int e^u \, du =$ $-\frac{1}{5} e^u + C =$ $-\frac{1}{5} e^{\frac{5}{x}} + C$

REVIEW SHEET # 4

1. $\int 2^{\sin x} \cos x \, dx$

2. $\int (\sin x)^2 \cos x \, dx$

3. $\int \sin^2 x \cos^2 x \, dx$

4. $\int 2^x \cos x \, dx$

5. $\int x \cos(x^2) \, dx$

6. $\int x \cos x \, dx$

7. $\int \sqrt{5}^{\tan x} \sec^2 x \, dx$

8. $\int (\tan x)^{\sqrt{5}} \sec^2 x \, dx$

9. $\int e^x \cos(e^x) \, dx$

10. $\int e^x \cos x \, dx$

REVIEW SHEET # 4 - SOLUTIONS

Problem	Hint	Solutions
1. $\int 2^{\sin x} \cos x \, dx$	$\frac{d}{dx}[\sin x] = ?$	We use the u-substitution: $u = \sin x$ $du = \cos x \, dx$ Our integral then becomes $\int 2^u \, du = \frac{2^u}{\ln 2} + C =$ $\frac{2^{\sin x}}{\ln 2} + C$
2. $\int (\sin x)^2 \cos x \, dx$	same as above	Using the same substitution as in problem #1 , our integral becomes $\int u^2 \, du = \frac{u^3}{3} + C =$ $\frac{\sin^3 x}{3} + C$
3. $\int \sin^2 x \cos^2 x \, dx$	Even powers of sin and cos	$\int \frac{1}{2}(1-\cos x)\ \frac{1}{2}(1+\cos x) \, dx =$ $\int \frac{1}{4}(1 - \cos^2 2x) \, dx =$ $\frac{1}{4}\int \sin^2 2x \, dx =$ $\frac{1}{4}\int \frac{1}{2}(1 - \cos 4x) \, dx =$ $\frac{1}{8}\left(x - \frac{1}{4}\sin 4x\right) + C$

Problem	Hint	Solution
4. $\int 2^x \cos x \, dx$	Product !	Integration by parts: $u = 2^x$ $\quad dv = \cos x \, dx$ $du = 2^x \ln 2 \, dx$ $\quad v = \sin x$ $\int 2^x \cos x \, dx = 2^x \sin x - \ln 2 \int 2^x \sin x \, dx$ Using integration by parts again on this last integral: $u = 2^x$ $\quad dv = \sin x \, dx$ $du = 2^x \ln 2 \, dx$ $\quad v = -\cos x$ Our integral equals: $2^x \sin x - \ln 2 \, (-2^x \cos x + \ln 2 \int 2^x \cos x dx)$ $= 2^x \sin x + (\ln 2) 2^x \cos x - (\ln 2)^2 \int 2^x \cos x dx$ Combining this last integral with our original one yields $(1+(\ln 2)^2) \int 2^x \cos x \, dx = 2^x \sin x + (\ln 2) 2^x \cos x$ and $\int 2^x \cos x \, dx = \frac{2^x \sin x + (\ln 2) 2^x \cos x}{(1+(\ln 2)^2)} + C$
5. $\int x \cos(x^2) \, dx$	Does this product require integration by parts? $\frac{d}{dx}[x^2] = ?$	$\boxed{\frac{1}{2}} \int \cos(x^2) \boxed{2} x \, dx =$ $\frac{1}{2} \int \cos u \, du = \quad (u = x^2)$ $\frac{1}{2} \sin u + C =$ $\frac{1}{2} \sin (x^2) + C$

Problem	Hint	Solution
6. $\int x \cos x \, dx$	Does this product require integration by parts?	Yes. $u = x$, $dv = \cos x \, dx$; $du = dx$, $v = \sin x$ $\int x \cos x \, dx = x \sin x - \int \sin x \, dx$ $= x \sin x + \cos x + C$
7. $\int \sqrt{5}^{\tan x} \sec^2 x \, dx$	$\frac{d}{dx}[\tan x] = ?$	Using the u-substitution $u = \tan x$: Our integral becomes: $\int \sqrt{5}^{u} \, du = \frac{\sqrt{5}^{u}}{\ln \sqrt{5}} + C =$ $\frac{\sqrt{5}^{\tan x}}{\ln \sqrt{5}} + C$
8. $\int (\tan x)^{\sqrt{5}} \sec^2 x \, dx$	Same as in #7	Using the same u-substitution as in # 7, our integral becomes $\int u^{\sqrt{5}} \, du = \frac{u^{\sqrt{5}+1}}{\sqrt{5}+1} + C$ $= \frac{(\tan x)^{\sqrt{5}+1}}{\sqrt{5}+1} + C$
9. $\int e^x \cos(e^x) \, dx$	Does this product require integration by parts?	Let $u = e^x$. Our integral becomes $\int \cos u \, du = \sin u + C =$ $\sin(e^x) + C$
10. $\int e^x \cos x \, dx$	Does this product require integration by parts?	Yes. $u = e^x$, $dv = \cos x \, dx$; $du = e^x \, dx$, $v = \sin x$ $\int e^x \cos x = e^x \sin x - \int e^x \sin x \, dx$ Repeating on the latter integral: $u = e^x$, $dv = \sin x \, dx$; $du = e^x \, dx$, $v = -\cos x$ our integral becomes: $e^x \sin x - (-e^x \cos x + \int e^x \cos x \, dx) =$ $e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$ Combine this last integral with our original one and divide by 2 to yield: $\frac{1}{2}(e^x \sin x + e^x \cos x) + C$

REVIEW SHEET # 5

1. $\int \tan^{-1} x \, dx$

2. $\int \frac{(\tan^{-1} x)^{10}}{1 + x^2} \, dx$

3. $\int \frac{dx}{1 + x^2}$

4. $\int \sin x \cos x \, dx$

5. $\int \sin 3x \cos x \, dx$

6. $\int \cos(\ln x) \frac{1}{x} \, dx$

7. $\int \cos(\ln x) \, dx$

8. $\int \ln(8x) \, dx$

9. $\int \frac{1}{x} \ln(8x) \, dx$

REVIEW SHEET # 5 - SOLUTIONS

Problem	Hint	Solution
1. $\int \tan^{-1}x\,dx$	$\tan^{-1}x = \tan^{-1}x \cdot 1$	Integration by parts: $u = \tan^{-1}x \qquad dv = dx$ $du = \frac{1}{1+x^2}\,dx \qquad v = x$ $\int \tan^{-1}x\,dx =$ $x\tan^{-1}x - \int \frac{x}{1+x^2}\,dx =$ $x\tan^{-1}x - \frac{1}{2}\ln\left\|1+x^2\right\| + C$
2. $\int \frac{(\tan^{-1}x)^{10}}{1+x^2}\,dx$	$\frac{d}{dx}\left[\tan^{-1}x\right] = ?$	We can use the u-substitution: $u = \tan^{-1}x$ $du = \frac{1}{1+x^2}\,dx$ which gives $\int u^{10}\,du = \frac{1}{11}u^{11} + C =$ $\frac{1}{11}(\tan^{-1}x)^{11} + C$
3. $\int \frac{dx}{1+x^2}$	Recognize it!	$\int \frac{dx}{1+x^2} = \tan^{-1}x + C$
4. $\int \sin x \cos x\,dx$	$\frac{d}{dx}(\sin x) = ?$	Let $u = \sin x$ $du = \cos x\,dx$. Our integral becomes: $\int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}\sin^2 x + C$

Problem	Hint	Solution
5. $\int \sin 3x \cos x \, dx$	$\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$	$\int \sin 3x \cos x \, dx =$ $\frac{1}{2}\int(\sin 4x + \sin 2x)\, dx =$ $\frac{1}{2}\left(-\frac{1}{4}\cos 4x - \frac{1}{2}\cos 2x\right) + C =$ $-\frac{1}{8}\cos 4x - \frac{1}{4}\cos 2x + C$
6. $\int \cos(\ln x)\,\frac{1}{x}\,dx$	$\frac{d}{dx}\left[\ln x\right] = ?$	Let $u = \ln x$, $du = \frac{1}{x}dx$. Our integral is $\int \cos u \, du = \sin u + C =$ $\sin(\ln x) + C$
7. $\int \cos(\ln x)\, dx$	Integration by parts	Let $u = \cos(\ln x)$, $dv = dx$; $du = -\sin(\ln x)\frac{1}{x}dx$, $v = x$. Our integral becomes $x\cos(\ln x) + \int \sin(\ln x)\, dx$. Again: $u = \sin(\ln x)$, $dv = dx$; $du = \cos(\ln x)\frac{1}{x}dx$, $v = x$. yielding: $x\cos(\ln x) + x\sin(\ln x) - \int \cos(\ln x)\, dx$. Combining this last integral with our original one yields $2\int \cos(\ln x)\, dx = x\cos(\ln x) + x\sin(\ln x)$ or $\int \cos(\ln x)\, dx = \frac{1}{2}\left[x\cos(\ln x) + x\sin(\ln x)\right] + C$

Problem	Hint	Solution
8. $\int \ln(8x)\,dx$	Integration by parts	Let $u = \ln(8x)$ $\quad dv = dx$ $du = \frac{1}{8x}\,8\,dx$ $= \frac{1}{x}\,dx$ $\quad v = x$ $\int \ln(8x)\,dx =$ $x\ln(8x) - \int 1\,dx =$ $x\ln(8x) - x + C$
9. $\int \frac{1}{x}\ln(8x)\,dx$	Does this one require integration by parts? $\frac{d}{dx}\left[\ln(8x)\right] = ?$	Using the u-substitution $u = \ln(8x)$ $du = \frac{1}{x}\,dx$, we have $\int u\,du = \frac{1}{2}u^2 + C =$ $\frac{1}{2}(\ln(8x))^2 + C$

REVIEW SHEET # 6

1. $\int \frac{dx}{x^2+4x+5}$

2. $\int \frac{dx}{x^2+4x-5}$

3. $\int \frac{dx}{(x^2+4x-5)^2}$

4. $\int \frac{x\,dx}{(x^2+16)^2}$

5. $\int \frac{x^2\,dx}{(x^2+16)^2}$

6. $\int \frac{2x^3+9x}{x^4+9x^2}\,dx$

7. $\int \frac{2x+6}{x^4+9x^2}\,dx$

REVIEW SHEET # 6 - SOLUTIONS

Problem

1. $\int \frac{dx}{x^2+4x+5}$

Hint

Is the denominator an irreducible quadratic?

Solution

Yes. We complete the square in the denominator to get

$$\int \frac{dx}{(x+2)^2+1} = \int \frac{du}{u^2+1} \qquad \begin{vmatrix} u = x+2 \\ du = dx \end{vmatrix}$$

$$= \tan^{-1} u + C = \tan^{-1}(x+2) + C$$

---

2. $\int \frac{dx}{x^2+4x-5}$

Is the denominator an irreducible quadratic?

No. We factor the denominator and use Partial Fractions:

$$\frac{1}{(x+5)(x-1)} = \frac{A}{x+5} \quad \frac{B}{x-1}$$

or

$$1 = A(x-1) + B(x+5) \qquad (**)$$

Letting $x = 1$ in (**) yields

$$1 = 6B \text{ or } B = \frac{1}{6}$$

Letting $x = -5$ in (**) yields

$$1 = -6A \text{ or } A = -\frac{1}{6}$$

so our integral becomes

$$-\frac{1}{6}\int \frac{dx}{x+5} + \frac{1}{6}\int \frac{dx}{x-1} =$$

$$-\frac{1}{6} \ln|x+5| + \frac{1}{6} \ln|x-1| + C$$

$$= \frac{1}{6} \ln \left| \frac{x-1}{x+5} \right| + C$$

Problem	Hint	Solution
3. $\int \frac{dx}{(x^2+4x-5)^2}$	See previous problem	Again, we use Partial Fractions:

$$1=A(x+5)(x-1)^2+B(x-1)^2+C(x-1)(x+5)^2 + D(x+5)^2 \qquad (***)$$

or, multiplying out:

$$1 = (A+C)x^3 + (3A+B+9C+D)x^2 + (-9A-2B+15C+10D)x + (5A+B-25C+25D)$$

Equating coefficients yields

$$\begin{cases} A + C = 0 \\ 3A + B + 9C + D = 0 \\ -9A - 2B + 15C + 10D = 0 \\ 5A + B - 25C + 25D = 1 \end{cases}$$

To help us solve these equations, we let $x = 1$ in (***) yielding

$$1 = 36D \quad \text{or} \quad D = \frac{1}{36}$$

Letting $x = -5$ in (***) yields

$$1 = 36B \quad \text{or} \quad B = \frac{1}{36}$$

Substituting these values into the second of our four equations:

$$3A + \frac{1}{36} + 9C + \frac{1}{36} = 0 \quad \text{or}$$

$$\begin{cases} 3A + 9C = -\frac{1}{18} \\ A + C + 0 \end{cases} \qquad \text{(first equation)}$$

which yields $A = -C = \frac{1}{108}$

so our integral becomes

(continued)

Problem	Hint	Solution
3. (continued)		$-\frac{1}{108}\int\frac{dx}{x+5} + \frac{1}{36}\int(x+5)^{-2}\,dx +$ $+\frac{1}{108}\int\frac{dx}{x-1} + \frac{1}{36}\int(x-1)^{-2}\,dx =$ $-\frac{1}{108}\ln\lvert x+5\rvert - \frac{1}{36}(x+5)^{-1} +$ $+\frac{1}{108}\ln\lvert x-1\rvert - \frac{1}{36}(x-1)^{-1} + C$
4. $\int\frac{x}{(x^2+16)^2}\,dx$	$\frac{d}{dx}(x^2+16) = ?$	We use the Power Rule here: $\boxed{\frac{1}{2}}\int(x^2+16)^{-2}\boxed{2}\,x\,dx =$ $\frac{1}{2}(-1)(x^2+16)^{-1} + C =$ $\frac{-1}{2}\cdot\frac{1}{x^2+16} + C$
5. $\int\frac{x^2}{(x^2+16)^2}dx$	Trig. Substitution	Let $x = 4\tan\theta$ $\quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$ $dx = 4\sec^2\theta\,d\theta$ $(x^2+16)^2 = (16\tan^2\theta+16)^2 =$ $(16(\tan^2\theta+1))^2 = (16\sec^2\theta)^2$ So our integral becomes $\int\frac{(16\tan^2\theta)4\sec^2\theta\,d\theta}{256\sec^4\theta} =$ $\frac{1}{4}\int\sin^2\theta\,d\theta = \frac{1}{4}\int\frac{1}{2}(1-\cos2\theta)d\theta$ $= \frac{1}{8}\left[\theta - \frac{1}{2}\sin 2\theta\right] + C =$ (continued)

Problem	Hint	Solution
5. (continued)		$\frac{1}{8}\theta - \frac{1}{16}\sin 2\theta + C =$ $\frac{1}{8}\theta - \frac{1}{16}(2)\sin\theta\cos\theta + C =$ $\frac{1}{8}\theta - \frac{1}{8}\sin\theta\cos\theta + C =$ $\frac{1}{8}\tan^{-1}\frac{x}{4} - \frac{1}{8}\frac{x}{\sqrt{x^2+16}}\frac{4}{\sqrt{x^2+16}} + C$ $= \frac{1}{8}\tan^{-1}\frac{x}{4} - \frac{1}{2}\frac{x}{x^2+16} + C$
6. $\int \frac{2x^3+9x}{x^4+9x^2}\,dx$	$\frac{d}{dx}(x^4+9x^2)=?$	Power Rule with n = -1: $\boxed{\frac{1}{2}}\int \frac{\boxed{2}(2x^3+9x)\,dx}{x^4+9x^2} =$ $\frac{1}{2}\ln\left\|x^4+9x^2\right\| + C$
7. $\int \frac{2x+6\ dx}{x^4+9x^2}$	Factor the denominator	We use Partial Fractions: $\frac{2x+6}{x^2(x^2+9)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+9}$ or $2x+6=Ax(x^2+9)+B(x^2+9)+(Cx+D)x^2$ (*) or $2x+6 = (A+C)x^3+(B+D)x^2+(9A)x+9B$ (continued)

Problem	Hint	Solution
7. (continued)		Equating coefficients, we have: $A + C = 0$ $B + D = 0$ $9A = 2$ $9B = 6$ which yields: $A = \frac{2}{9}$, $B = \frac{2}{3}$, $C = -\frac{2}{9}$, $D = -\frac{2}{3}$ so our integral becomes $\frac{2}{9}\int\frac{1}{x}\,dx + \frac{2}{3}\int x^{-2}\,dx - \frac{1}{9}\int\frac{2x}{x^2+9} - \frac{2}{3}\int\frac{dx}{x^2+9} =$ $\frac{2}{9}\ln\lvert x\rvert - \frac{2}{3}x^{-1} - \left(\frac{1}{9}\right)\ln\lvert x^2+9\rvert - \frac{2}{9}\tan^{-1}\frac{x}{3} + C$

REVIEW SHEET # 7

1. $\int \sin^{19} 2x \cos 2x \, dx$

2. $\int \sin^{19} 2x \cos^{3} 2x \, dx$

3. $\int \sin^{5} \frac{1}{8}x \, dx$

4. $\int \cos^{4} x \sin^{3} x \, dx$

5. $\int \sin^{4} x \, dx$

6. $\int \tan^{100} \pi x \sec^{2} \pi x \, dx$

7. $\int \tan^{100} \pi x \sec^{4} \pi x \, dx$

8. $\int \tan x \sec x \, dx$

9. $\int \tan^{3} x \sec^{28} x \, dx$

10. $\int \tan^{4} 5x \, dx$

11. $\int \sec^{4} x \, dx$

12. $\int \sec^{3} x \, dx$

REVIEW SHEET #7 - SOLUTIONS

Problem	Hint	Solution
1. $\int \sin^{19} 2x \cos 2x \, dx$	$\frac{d}{dx}[\sin 2x] = ?$	Using the power rule (n = 19) $\boxed{\frac{1}{2}}\int (\sin 2x)^{19} \boxed{2} \cos 2x \, dx =$ $\frac{1}{2} \cdot \frac{1}{20} (\sin 2x)^{20} + C =$ $\frac{1}{40} \sin^{20} x + C$
2. $\int \sin^{19} 2x \cos^{3} 2x dx$	odd power of cos $\cos^2 2x = 1 - \sin^2 2x$	$\int \sin^{19} 2x \cos^2 2x \cos 2x \, dx =$ $\int \sin^{19} 2x (1 - \sin^2 2x) \cos 2x \, dx =$ $\int (\sin^{19} 2x - \sin^{21} 2x) \cos 2x \, dx =$ $\boxed{\frac{1}{2}} \int [(\sin 2x)^{19} - (\sin 2x)^{21}] \boxed{2} \cos 2x dx =$ $\frac{1}{2}\left[\frac{1}{20}(\sin 2x)^{20} - \frac{1}{22}(\sin 2x)^{22}\right] + C$ $= \frac{1}{40} \sin^{20} 2x - \frac{1}{44} \sin^{22} 2x + C$
3. $\int \sin^{5} \frac{1}{8} x \, dx$	odd power of sin $\sin^2 \frac{1}{8}x = 1 - \cos^2 \frac{1}{8}x$	$\int \sin^4 \frac{1}{8}x \sin \frac{1}{8}x \, dx =$ $\int (\sin^2 \frac{1}{8}x)^2 \sin \frac{1}{8}x \, dx =$ $\int (1 - \cos^2 \frac{1}{8}x)^2 \sin \frac{1}{8}x \, dx =$ $\int (1 - 2\cos^2 \frac{1}{8}x + \cos^4 \frac{1}{8}x) \sin \frac{1}{8}x \, dx =$ $\boxed{-8}\int (1 - 2\cos^2 \frac{1}{8}x + \cos^4 \frac{1}{8}x) \boxed{\frac{-1}{8}} \sin \frac{1}{8}x \, dx =$ $-8 (\cos \frac{1}{8}x - \frac{2}{3}\cos^3 \frac{1}{8}x + \frac{1}{5}\cos^5 \frac{1}{8}x) + C$

Problem	Hint	Solution
4. $\int \cos^4 x \sin^3 x \, dx$	odd power of sin $\sin^2 x = 1 - \cos^2 x$	$\int \cos^4 x \sin^2 x \sin x \, dx =$ $\int \cos^4 x (1 - \cos^2 x) \sin x \, dx =$ $\int (\cos^4 x - \cos^6 x) \sin x \, dx =$ $\boxed{-1} \int (\cos^4 x - \cos^6 x) \boxed{-1} \sin x \, dx =$ $-\left(\frac{1}{5}\cos^5 x - \frac{1}{7}\cos^7 x\right) + C =$ $-\frac{1}{5}\cos^5 x + \frac{1}{7}\cos^7 x + C$
5. $\int \sin^4 x \, dx$	Even powers only of sin or cos Half angle formula: $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$	$\int \sin^2 x \sin^2 x \, dx =$ $\int \frac{1}{2}(1 - \cos 2x) \frac{1}{2}(1 - \cos 2x) \, dx =$ $\frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x) \, dx =$ $\frac{1}{4}\int 1 \, dx - \frac{1}{2}\int \cos 2x \, dx + \frac{1}{4}\int \cos^2 2x \, dx$ $= \frac{1}{4}x - \frac{1}{2}\cdot\frac{1}{2}\sin 2x + \frac{1}{4}\int \frac{1}{2}(1 + \cos 4x) \, dx$ $= \frac{1}{4}x - \frac{1}{4}\sin 2x + \frac{1}{8}\left(x + \frac{1}{4}\sin 4x\right) + C$ $= \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$
6. $\int \tan^{100} \pi x \sec^2 \pi x \, dx$	$\frac{d}{dx}[\tan \pi x] = ?$	$\boxed{\frac{1}{\pi}} \int (\tan \pi x)^{100} \boxed{\pi} \sec^2 \pi x \, dx =$ $\frac{1}{\pi}\cdot\frac{1}{101}\tan^{101} \pi x + C$

Problem	Hint	Solution
7. $\int \tan^{100} \pi x \sec^{4} \pi x\, dx$	Even power of sec $\sec^2 \pi x = \tan^2 \pi x + 1$	$\int \tan^{100} \pi x \sec^{2} \pi x \sec^{2} \pi x\, dx =$ $\int \tan^{100} \pi x (\tan^{2} \pi x + 1) \sec^{2} \pi x\, dx =$ $\int (\tan^{102} \pi x + \tan^{100} \pi x) \sec^{2} \pi x\, dx =$ $\boxed{\frac{1}{\pi}} \int \tan^{102} \pi x + \tan^{100} \pi x) \boxed{\pi} \sec^{2} \pi x\, dx =$ $\frac{1}{\pi} \left(\frac{1}{103} \tan^{103} \pi x + \frac{1}{101} \tan^{101} \pi x \right) + C$
8. $\int \tan x \sec x\, dx$	$\frac{d}{dx}(?) = \tan x \sec x$	$\int \tan x \sec x\, dx = \sec x + C$
9. $\int \tan^{3} x \sec^{28} x\, dx$	odd power of tan $\tan^2 x = \sec^2 x - 1$	$\int \tan^{2} x \sec^{27} x \sec x \tan x\, dx =$ $\int (\sec^{2} x - 1) \sec^{27} x \sec x \tan x\, dx =$ $\int (\sec^{29} x - \sec^{27} x) \sec x \tan x\, dx =$ $\frac{1}{30} \sec^{30} x - \frac{1}{28} \sec^{28} x + C$
10. $\int \tan^{4} 5x\, dx$	$\tan^2 5x = \sec^2 5x - 1$	$\int \tan^{2} 5x \tan^{2} 5x\, dx =$ $\int \tan^{2} 5x (\sec^{2} 5x - 1)\, dx =$ $\int \tan^{2} 5x \sec^{2} 5x\, dx - \int \tan^{2} 5x\, dx =$ $\boxed{\frac{1}{5}} \int (\tan 5x)^{2} \boxed{5} \sec^{2} 5x\, dx -$ $\int (\sec^{2} 5x - 1)\, dx =$ $\frac{1}{5} \cdot \frac{1}{3} \tan^{3} 5x - \int \sec^{2} 5x\, dx + \int 1\, dx =$ $\frac{1}{15} \tan^{3} 5x - \frac{1}{5} \tan 5x + x + C$

Problem	Hint	Solution
11. $\int \sec^4 x\, dx$	Even power of sec $\sec^2 x = \tan^2 x + 1$	$\int \sec^2 x \sec^2 x\, dx =$ $\int (\tan^2 x + 1) \sec^2 x\, dx =$ $\frac{1}{3}\tan^3 x + \tan x + C$
12. $\int \sec^3 x\, dx$	odd power of sec integration by parts	Let $u = \sec x$, $dv = \sec^2 x\, dx$ $du = \sec x \tan x\, dx$, $v = \tan x$ $\int \sec^3 x\, dx =$ $\sec x \tan x - \int \tan^2 x \sec x\, dx =$ $\sec x \tan x - \int (\sec^2 x - 1)\sec x\, dx =$ $\sec x \tan x + \int \sec x\, dx - \int \sec^3 x\, dx =$ $\sec x \tan x + \ln \lvert \sec x + \tan x \rvert - \int \sec^3 x\, dx$ Combining this last integral with the original integral yields $2\int \sec^3 x\, dx = \sec x \tan x + \ln \lvert \sec x + \tan x \rvert$ so that $\int \sec^3 x\, dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln \lvert \sec x + \tan x \rvert + C$